Generalized optical theorem for surface waves and layered media

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We present a generalized optical theorem for surface waves. The theorem also applies to body waves since under many circumstances body waves can be written in terms of surface-wave modal summations. This theorem therefore extends the domain of applicability of the optical theorem from homogeneous background media to a general class of body and surface-wave propagation regimes within layered elastic media.

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I. INTRODUCTION

The optical theorem uses the conservation of energy to relate the energy radiated from a scattering body to the amplitude decay of the wave that was incident upon the scatterer (e.g., due to backscattering). For homogeneous background media, it is possible to formulate a generalized optical theorem that correctly describes the conservation of energy in a general class of scattering problems (for both acoustic and elastic waves). A similar generalized optical theorem that accounts for the scattering of multimode surface waves would extend the applicability of the optical theorem to account for vertical heterogeneity in the background medium. In this paper, we derive such an optical theorem.

By a generalized optical theorem, we refer to an optical theorem which gives an integral condition on the scattering amplitude for any specific angle of incidence and any scattering angle. From this generalized optical theorem, other relationships can then be derived which describe scattering relationships for more specific forms of scattering. The generalized theorem for acoustic waves has been derived in different ways by many authors. For example, Newton [1] gave an account of Heisenberg's use of the unitarity properties of the scattering matrix in order to derive the generalized optical theorem. Glauber and Schomaker [2] used reciprocity relations to show the reversibility of the scattering amplitude between any pair of directions and further to derive the generalized optical theorem. They then derive more specific optical theorems for forward scattering (when the angle of incidence equals the scattering angle) and for scattering with inversion symmetry. Marston [3] used a similar approach using symmetry, reciprocity, and energy conservation to derive the same result for acoustic scattering with inversion symmetry. Representation theorems (or reciprocity relations) have also been used extensively to study energy relations in scattering problems [4–6] and Snieder *et al.* [7] presented an alternative derivation of the generalized optical theorem using an approach based on the use of the interferometric Green's function representations (specific forms of representation theorems [8,9]). Further, Budreck and Rose [10] derived a generalized optical theorem for elastodynamics using elastodynamic scattering theory and a Newton-Marchenko equation.

Optical theorems find a wide range of applications in physics including testing of algorithms for the computation of scattered wave fields [11,12], the estimation of backscattering from measurements of the scattered wave field taken at other angles [3], determining phase shifts from the measurement of scattering data (e.g., in quantum mechanics [13,14]), the investigation of the attenuation effect of scatterers (e.g., in acoustics [15], and in seismology [16,17]), the determination of the energy both scattered and absorbed by a scatterer (in acoustics [4]), and by using a statistical approach it may be possible to infer the structure of the scattering media [18].

In this paper, we derive a generalized optical theorem for surface waves. The benefit of such a theorem over and above body wave optical theorems is that the surface-wave theory allows us to consider vector wave fields and multiple surface-wave modes [19,20]. In seismology, a surface-wave mode refers to a wave that propagates laterally across the surface of the Earth and exists due to the presence of the free surface. In a homogeneous half space, only one mode exists (the fundamental Rayleigh mode). However, if the medium of interest is vertically heterogeneous then so-called higher-mode Rayleigh waves (and fundamental and higher-mode Love waves) exist; all propagating with different frequency-dependent phase velocities [21]. Therefore the generalized optical theorem for surface waves derived here enables the range of applications of the generalized optical theorem to be extended to cases where surface waves are produced or where media may be represented as layered. This includes seismology [20], quantum physics [22], acoustoelectrics [23], and materials science [24]. Since body waves can also be represented by a sum over many surface-wave modes [25,26], the optical theorem for surface waves extends to a general class of body and surface-wave propagation regimes within layered elastic media.

We derive the optical theorem by considering the interferometric Green's function representations for elastic media and using the appropriate scattered surface-wave Green's functions. In places our approach mirrors that of Snieder *et al.* [7]. However, while those authors consider scalar acoustic wave fields propagating in homogeneous media, our approach uses the surface-wave Green's functions for wave

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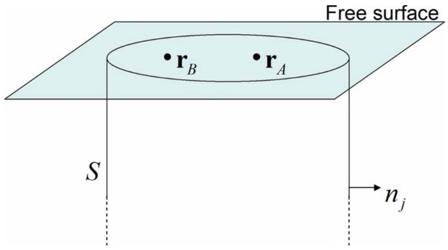

FIG. 1. (Color online) Geometry for Eq. (1). Note that $\mathbf{r}_A$ and $\mathbf{r}_B$ lie beneath the free surface in this case.

propagation in elastic media. Hence, we derive the first such theorem for vector wave fields in layered elastic media.

We first define the appropriate interferometric Green's function representation: such representations relate the Green's functions between two points within a bounding surface to the Green's functions between the bounding surface and each of the two points [9,27,28]. Second, we define the appropriate Green's functions to describe a single-scattered surface-wave field and insert those into the interferometric representation. This results in four different contributing terms that can be analyzed using a stationary-phase approach (where we assume that the dominant contribution of each integral term comes from the point on the integration surface at which the phase of the integrand is stationary). We find that in order for the interferometric representation to hold, two sets of nonphysical terms must cancel; this condition results in a generalized optical theorem for surface waves. While we consider only Rayleigh surface waves, an identical analysis exists for Love waves.

II. INTERFEROMETRIC GREEN'S FUNCTION REPRESENTATIONS

Interferometric Green's function representations can be derived from representation theorems by using appropriate mathematical representations of the Green's functions between two locations ($\mathbf{r}_A$ and $\mathbf{r}_B$) and between each of those locations and all points on a bounding surface S (Fig. 1). S may be arbitrarily shaped, but in the following we consider the specific case of a cylinder extending to great depth. In this paper, we use semianalytical representations of the particle-displacement surface-wave Green's functions; hence we consider an integral describing the extraction of particle-displacement point-force source Green's functions in elastic media [27],

$$G^*_{im}(\mathbf{r}_B,\mathbf{r}_A) - G_{im}(\mathbf{r}_B,\mathbf{r}_A) = \int_{\mathbf{r}_S\in S}\{G_{in}(\mathbf{r}_B,\mathbf{r}_S)n_j c_{njkl}\partial_k G^*_{ml}(\mathbf{r}_A,\mathbf{r}_S) - n_j c_{njkl}\partial_k G_{il}(\mathbf{r}_B,\mathbf{r}_S)G^*_{mn}(\mathbf{r}_A,\mathbf{r}_S)\}dS, \quad (1)$$

where $G_{im}(\mathbf{r}_B,\mathbf{r}_A)$ denotes the Green's function representing the ith component of particle displacement at location $\mathbf{r}_B$ due to a unidirectional impulsive point force in the m direction at $\mathbf{r}_A$, $\partial_k G_{ml}(\mathbf{r}_A,\mathbf{r}_S)$ is the spatial partial derivative at location $\mathbf{r}_S$ taken in the k direction of the Green's function $G_{ml}(\mathbf{r}_A,\mathbf{r}_S)$,

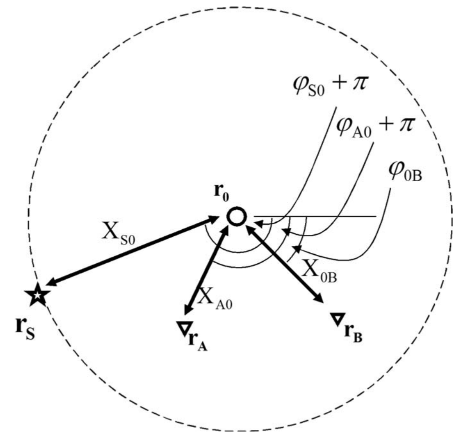

FIG. 2. Sketch illustrating the geometry in the horizontal plane that is used in the stationary-phase analysis. The scatterer $\mathbf{r}_0$ is placed at the center of the coordinate system ($\mathbf{r}=\mathbf{0}$).

c_{njkl} is the elasticity tensor, superscript * denotes complex conjugation, and n_j is the outward normal to the arbitrarily shaped closed surface S, where S encloses the locations $\mathbf{r}_A$ and $\mathbf{r}_B$ (Fig. 1). Einstein's summation convention applies for repeat indices. The term $n_j c_{njkl}\partial_k G_{ml}(\mathbf{r}_A,\mathbf{r}_S)$ represents the particle displacement at $\mathbf{r}_A$ due to a deformation-rate tensor source at $\mathbf{r}_S$. In seismology, integrals, such as Eq. (1), are commonly used to extract the inter-receiver Green's function estimates from recordings of seismic wave fields at each receiver; a process referred to as seismic interferometry [29].

III. GREEN'S FUNCTIONS FOR SURFACE-WAVE PROPAGATION

In order to solve Eq. (1) for scattered surface waves, we require an appropriate coordinate system and appropriate forms for the Green's functions. In order to solve the interferometric integral, we use a cylindrical coordinate system with the scatterer placed at radius equal to zero and define the locations $\mathbf{r}_A$, $\mathbf{r}_B$, $\mathbf{r}_S$, and $\mathbf{r}_0$ as (Fig. 2)

$$\mathbf{r}_A = \begin{pmatrix} X_{A0}\cos(\varphi_{A0}+\pi) \\ X_{A0}\sin(\varphi_{A0}+\pi) \\ z_A \end{pmatrix}, \quad \mathbf{r}_B = \begin{pmatrix} X_{0B}\cos\varphi_{0B} \\ X_{0B}\sin\varphi_{0B} \\ z_B \end{pmatrix},$$

$$\mathbf{r}_S = \begin{pmatrix} X_{S0}\cos(\varphi_{S0}+\pi) \\ X_{S0}\sin(\varphi_{S0}+\pi) \\ z_S \end{pmatrix}, \quad \mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix}. \quad (2)$$

In the Green's functions that we introduce in Appendix A, the terms such as X_{A0} and φ_{A0} describe the propagation path of the surface wave. The order of the subscripts identifies the direction of propagation, for example, $A0$ denotes that these parameters describe the wave propagating from $\mathbf{r}_A$ to $\mathbf{r}_0$. For consistency, we have defined the vector (2) using the same notation as Appendix A. The cylindrical coordinate system is centered on the scatterer and this requires that for the angles describing propagation *toward* the scatterer, we must add a factor π since all vectors are defined pointing *away* from the scatterer.

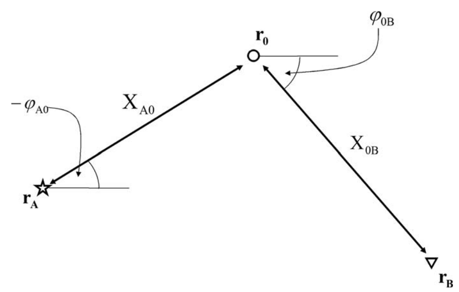

FIG. 3. Geometric variables used to describe the scattered surface wave propagating between $\mathbf{r}_A$ and $\mathbf{r}_B$.

In our analysis, we assume a single incident surface-wave mode (ν) and a single-scattered surface-wave mode (σ); to simplify the notation we define the partial Green's function (herein referred to as the Green's function) representing the combination of these two modes. To represent the *full* Green's function, we would require a sum over all the partial Green's functions, hence over all incoming and outgoing modes.

The Green's function representing the particle displacement due to a point force is the sum of the incident and scattered wave fields. For notational convenience, we drop superscripts $\sigma\nu$ as follows:

$$G_{im}^{\sigma\nu}(\mathbf{r}_B,\mathbf{r}_A) = G_{im}^{0}(\mathbf{r}_B,\mathbf{r}_A) + G_{im}^{sc}(\mathbf{r}_B,\mathbf{r}_A), \tag{3}$$

and the equivalent particle-displacement deformation-rate Green's function is

$$n_j c_{njkm}\partial_k G_{im}^{\sigma\nu}(\mathbf{r}_B,\mathbf{r}_A) = \partial G_{im}^{0}(\mathbf{r}_B,\mathbf{r}_A) + \partial G_{im}^{sc}(\mathbf{r}_B,\mathbf{r}_A), \tag{4}$$

where $G_{im}^{0}(\mathbf{r}_B,\mathbf{r}_A)$ and $\partial G_{im}^{0}(\mathbf{r}_B,\mathbf{r}_A)$ represent the direct waves observed at $\mathbf{r}_B$ due to a unidirectional point force at $\mathbf{r}_A$ and a deformation-rate tensor source at $\mathbf{r}_A$, respectively, and $G_{im}^{sc}(\mathbf{r}_B,\mathbf{r}_A)$ and $\partial G_{im}^{sc}(\mathbf{r}_B,\mathbf{r}_A)$ are the corresponding terms for the scattered wave field. These terms are defined in detail in Appendix A, with appropriate geometrical variables illustrated in Figs. 3 and 4.

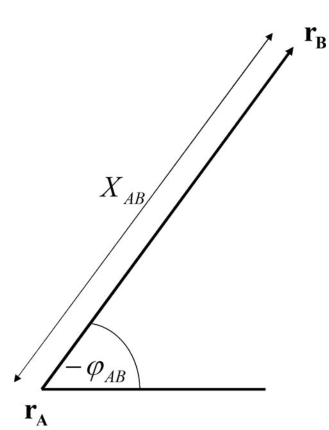

FIG. 4. Geometric variables used to describe the direct surface wave propagating between $\mathbf{r}_A$ and $\mathbf{r}_B$.

IV. SOLUTION OF THE INTERFEROMETRIC REPRESENTATION

For Eqs. (3) and (4) to solve the interferometric representation successfully, we require that when the right-hand side of Eq. (1) is evaluated using the Green's functions (3) and (4) we obtain the Green's function of the same form [as defined by the left-hand side of Eq. (1)].

To evaluate the right-hand side, we must solve the integral over the surface S (Fig. 1). To do so we use the method of stationary phase. This method has been shown to be a valuable tool to analyze and understand the application of seismic interferometry in various settings [30–32]. With a stationary-phase analysis we make a high-frequency approximation and assume that the dominant contributions to the interferometric integral come from those points at which the phase of the integrand is stationary [33]. We further assume that the amplitude of the integrand varies slowly around these stationary points.

To solve the interferometric integral, we substitute the Green's functions (3) and (4) into Eq. (1) resulting in four terms: the cross correlation of the direct Rayleigh wave at one receiver with the direct Rayleigh wave at the other ($T1$), the cross correlation of the direct Rayleigh wave at one receiver with the scattered surface wave at the other (and vice versa, $T2$ and $T3$), and the cross correlation of the scattered surface wave at one receiver with the scattered surface wave at the other ($T4$). We label our terms $T1$ to $T4$ to keep our notation consistent with previous work in seismology [32] and in acoustics [7]. In Appendix A we analyze each of these terms (for isotropic elastic media) using a stationary-phase analysis and find that each term contributes as follows.

$T1$: the stationary-phase analysis of this term is identical to that presented by Halliday and Curtis [30]. This results in the part of the Green's function corresponding to the direct surface wave (i.e., the wave field in the layered background medium). We can therefore write term $T1$ as

$$T1 = G_{im}^{0^*}(\mathbf{r}_B,\mathbf{r}_A) - G_{im}^{0}(\mathbf{r}_B,\mathbf{r}_A), \tag{5}$$

$T2$ and $T3$: In Appendix A we show that the stationary-phase analysis naturally divides these two terms into four separate subterms (the geometries for the stationary-phase analysis are illustrated in Fig. 5). Each of terms $T2$ and $T3$ contributes a first subterm that corresponds to part of the Green's function in Eq. (3). We refer to this contribution as the physical contribution [indicated by a subscript p, with geometries illustrated in Figs. 6(a) and 6(c)],

$$T2_p + T3_p = G_{im}^{sc^*}(\mathbf{r}_B,\mathbf{r}_A) - G_{im}^{sc}(\mathbf{r}_B,\mathbf{r}_A). \tag{6}$$

Hence, the correct (physical) scattered surface waves are recovered from terms $T2$ and $T3$. However, we find that the second subterms of each of $T2$ and $T3$ do not correspond to any part of the true Green's function (3) that we expect from the left-hand side of Eq. (1). We use subscript np to indicate that this is a nonphysical term and geometries are illustrated in Figs. 6(b) and 6(d) (note from hereon, Einstein's summation convention for repeat indices *does not* apply),

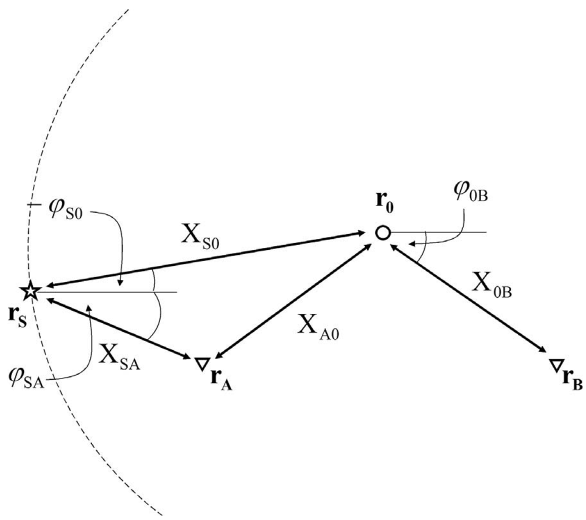

FIG. 5. Definition of geometric variables required for terms $T2$ and $T3$ in the horizontal plane. Here we show the geometry of the direct surface wave at receiver location $\mathbf{r}_A$ due to a source at location $\mathbf{r}_S$, the horizontal projection of the source-receiver path length is X_{SA}, and the horizontal projection of the azimuth is φ_{SA}. The scattered wave is shown between source location $\mathbf{r}_S$ and receiver location $\mathbf{r}_B$, with the scatterer located at $\mathbf{r}_0$. The horizontal projection of the source-scatterer path is defined by length X_{S0} and angle φ_{S0}, and the horizontal projection of the scatterer-receiver path is similarly defined by X_{0B} and φ_{0B}. Finally, we define the offset between $\mathbf{r}_A$ and $\mathbf{r}_0$ as X_{A0}.

$$T2_{np} + T3_{np} = -\frac{2}{\pi}A^{\sigma\nu}[S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A) - S_{im}^{\sigma\nu^*}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)]. \quad (7)$$

In Eq. (7), $A^{\sigma\nu}$ represents the propagation characteristics of the incident and scattered waves (i.e., the phase, wave number, and geometrical spreading) between the excitation and observation point, respectively. $S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)$ is the three-dimensional (3D) surface-wave scattering matrix for an incident surface-wave mode ν, traveling in the direction of the unit vector $\hat{\mathbf{r}}_A$, that is scattered in the direction of the unit vector $\hat{\mathbf{r}}_B$ as surface-wave mode σ, where it is understood that $\hat{\mathbf{r}}_A$ and $\hat{\mathbf{r}}_B$ are the horizontal components of the unit vectors. We refer to $S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)$ as the 3D scattering matrix as it includes polarization terms for the excitation of the incident wave field and for the observed scattered wave field, i.e.,

$$S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A) = P_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A) f^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A), \quad (8)$$

where $P_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)$ is the product of the polarization terms of the excited wave field (a point force in the m direction exciting mode ν) and the observed wave field (the i component of particle displacement of the scattered mode σ), and $f^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)$ is the surface-wave scattering matrix for an incident surface-wave mode ν, traveling in the direction of unit vector $\hat{\mathbf{r}}_A$, scattered in the direction of unit vector $\hat{\mathbf{r}}_B$ as the surface-wave mode σ. Since the polarization terms are dependent on the depth of excitation and observation, the scattering term $S_{im}^{\sigma\nu}$ is dependent on the depths of the scatterer and the points of excitation and observation.

$T4$: term $T4$ is the cross correlation of the scattered surface waves recorded at both receivers. In Appendix A we show that this can be written as

$$T4 = -\frac{2i}{\pi^2}A^{\sigma\sigma}\int_0^{2\pi}\left[\frac{1}{P_{im}^{\nu\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)}S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,-\hat{\mathbf{r}}_S) \times S_{im}^{\sigma\nu^*}(\hat{\mathbf{r}}_A,-\hat{\mathbf{r}}_S)\right]d\Omega, \quad (9)$$

where $P_{im}^{\nu\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)$ is the product of the polarization terms of the surface-wave mode ν traveling in the direction of the horizontal unit vectors $\hat{\mathbf{r}}_A$ and $\hat{\mathbf{r}}_B$, and $A^{\sigma\sigma}$ accounts for the propagation characteristics of the scattered waves observed at $\mathbf{r}_A$ and $\mathbf{r}_B$. The integration is over the azimuth of the incident wave upon the scatterer (i.e., over $-\hat{\mathbf{r}}_S$). The term $1/P_{im}^{\nu\nu}$ cancels the excitation terms for the incident wave field that appear in the product $S_{im}^{\sigma\nu}S_{im}^{\sigma\nu^*}$; hence the depths of the sources exciting the incident wave fields do not have an effect on expression (9).

V. GENERALIZED OPTICAL THEOREM FOR SURFACE WAVES

We have already shown that the correct direct and scattered surface waves are recovered from terms $T1$, $T2_p$, and $T3_p$. Note that from Eqs. (5) and (6) we can write,

$$T1 + T2_p + T3_p = [G_{im}^{0*}(\mathbf{r}_B,\mathbf{r}_A) + G_{im}^{sc*}(\mathbf{r}_B,\mathbf{r}_A)] - [G_{im}^{0}(\mathbf{r}_B,\mathbf{r}_A) + G_{im}^{sc}(\mathbf{r}_B,\mathbf{r}_A)], \quad (10)$$

$$= G_{im}^{*}(\mathbf{r}_B,\mathbf{r}_A) - G_{im}(\mathbf{r}_B,\mathbf{r}_A). \quad (11)$$

Thus, the combination of these terms satisfies the left-hand side of Eq. (1). However, since Eq. (1) is exact, the nonphysical arrivals introduced by terms $T2_{np}$, $T3_{np}$, and $T4$ must cancel. We therefore require that $T2_{np}+T3_{np}+T4=0$, which on expansion becomes,

$$\frac{2}{\pi}A^{\sigma\nu}[S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A) - S_{im}^{\sigma\nu^*}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)] = \frac{2}{i\pi^2}A^{\sigma\sigma}\int_0^{2\pi}\left[\frac{1}{P_{im}^{\nu\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)}S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,-\hat{\mathbf{r}}_S)S_{im}^{\sigma\nu^*}(\hat{\mathbf{r}}_A,-\hat{\mathbf{r}}_S)\right]d\Omega. \quad (12)$$

Finally we write Eq. (12) in the form

$$S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A) - S_{im}^{\sigma\nu^*}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A) = \frac{D^{\sigma\nu}}{i\pi}\int_0^{2\pi}\left[\frac{1}{P_{im}^{\nu\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)}S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,-\hat{\mathbf{r}}_S)S_{im}^{\sigma\nu^*}(\hat{\mathbf{r}}_A,-\hat{\mathbf{r}}_S)\right]d\Omega. \quad (13)$$

This is a generalized optical theorem for surface waves, describing the relationship between any incident surface-wave mode ν excited by any point-force component m and any scattered surface-wave mode σ observed as any particle-displacement component i. It has a slightly more complicated form in layered media due to the presence of multiple modes.

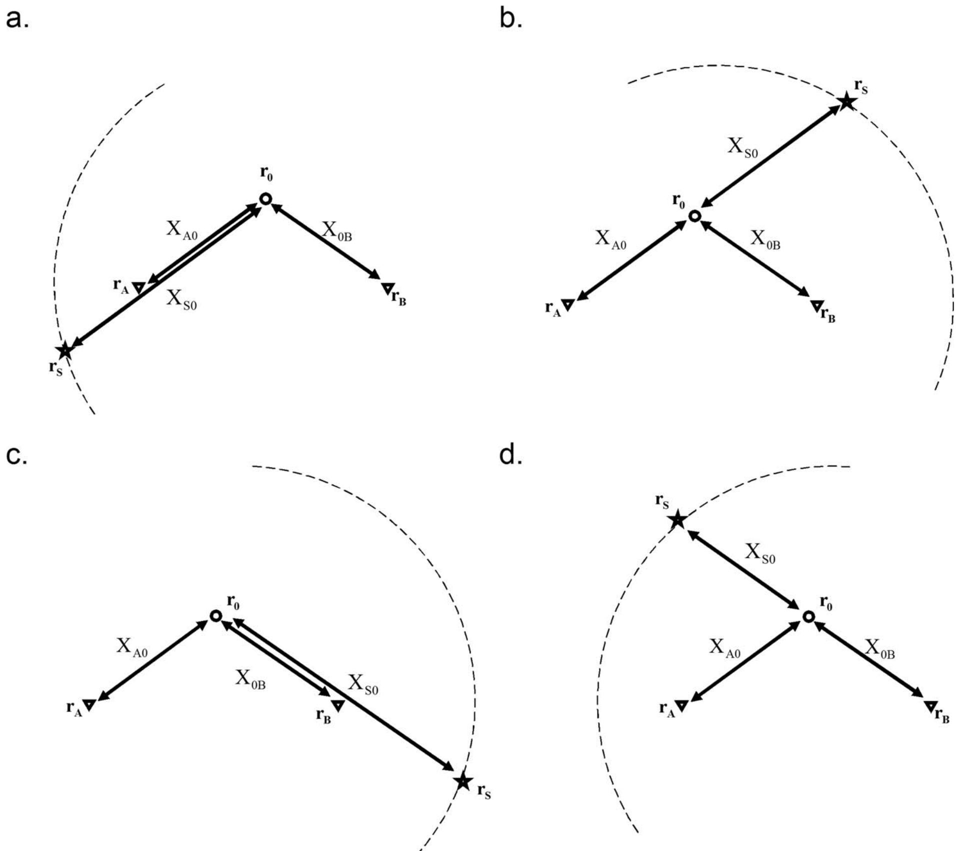

FIG. 6. There are four types of stationary point illustrated by boundary locations $\mathbf{r}_S$, relating to the recovery of a wave propagating from receiver $\mathbf{r}_A$ to receiver $\mathbf{r}_B$, scattered en route by a heterogeneity at $\mathbf{r}_0$. We use a circular boundary of sources for illustration (dashed line). (a) Term $T2$ (physical), (b) term $T2$ (nonphysical), (c) term $T3$ (physical), and (d) term $T3$ (nonphysical). To illustrate term $T3$, we have defined the additional geometrical term X_{SB} describing the horizontal offset along the path between the source $\mathbf{r}_S$ and receiver $\mathbf{r}_B$.

The term $D^{\sigma\nu}$ contains ratios of the phase and wave number of each of the modes σ and ν (i.e., it accounts for the fact that each mode has different propagation characteristics), and the term $P_{im}^{\nu\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)$ removes the polarization terms of the incident wave fields from the scattering terms $S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,-\hat{\mathbf{r}}_S)$ and $S_{im}^{\sigma\nu^*}(\hat{\mathbf{r}}_A,-\hat{\mathbf{r}}_S)$.

Note that in the case of a homogeneous half space $D^{\sigma\nu}=1$ (i.e., only one surface-wave mode is present). If we assume that the outgoing mode is observed with the same component as that with which the incoming mode was excited (i.e., $i=m$) then the polarization terms of the excited and observed wave fields implicit in $S_{im}^{\sigma\nu}$ cancel. This allows us to use only the scattering matrix f (dropping the superscripts) and with the reversibility of this scattering matrix,

$$f(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A) - f^*(\hat{\mathbf{r}}_A,\hat{\mathbf{r}}_B) = \frac{1}{i\pi}\int_0^{2\pi} f(\hat{\mathbf{r}}_B,-\hat{\mathbf{r}}_S)f^*(\hat{\mathbf{r}}_A,-\hat{\mathbf{r}}_S)d\Omega, \tag{14}$$

which resembles the previously derived generalized optical theorem.

VI. CONCLUSIONS

We have derived a generalized optical theorem for surface waves using the interferometric Green's function representation and semianalytical Green's functions for scattered surface waves. This analysis accounts for the scattering of higher-mode surface waves, and while our analysis uses the Rayleigh-wave Green's functions the results are equally applicable to Love-wave modes. Also note that while we consider isotropic elastic media it may be possible to use adaptations of the surface-wave theory to derive a similar relation for anisotropic media (see, for example, the discussion on surface-wave propagation in anisotropic layered media presented by Aki and Richards [21], chapter 7).

The ability to account for higher-mode surface waves means that this theorem can be applied to surface waves propagating in layered media, as this layering is manifest in the presence of such higher-mode surface waves. It has also been shown in previous studies that it is possible to represent a body wave field as a sum over surface-wave modes using the Green's functions such as those used here. For example, in a seismological study Nolet *et al.* [26] used a locked-mode approximation (the introduction of a total-internal reflector at some depth) to model the full wave field using the surface-wave Green's functions for vertically heterogeneous media. This approximation essentially turns the problem into one of elastic wave propagation in a closed layered medium. Hence the optical theorem derived here not only applies to surface waves in layered elastic half spaces but also to surface *and* body waves in closed layered elastic media. Note also that modal summations can be used to model the exact wave field in an open layered elastic half space using so-called leaky modes [25]. Hence, the theorem can be applied to a general class of body and surface-wave propagation regimes within layered elastic media.

This generalized optical theorem for surface waves compliments previous derivations of the generalized optical theorem for homogeneous background media [1–3,7] and previous work considering the role of (more specific versions of) the optical theorem on the attenuation of surface waves due to scattering [16,17]. The generalized optical theorem for surface waves that we derive here may allow the range of applications of the generalized optical theorem to be extended to those areas of physics where surface waves are observed, including quantum physics [22], material physics [24], seismology [20], and acoustoelectrics [23].

Finally, using the above method of derivation, it may be possible to obtain similar relationships for other modes of energy propagation or for different source and receiver quantities. For example, Wapenaar *et al.* [34] and Snieder *et al.* [35] derived generalized interferometric relations that include (among others) seismoelectric, electrokinetic, electromagnetic, and diffusion-phenomena Green's functions. With appropriate representations of these Green's functions, an interferometric approach may allow for the derivation of similar optical theorem-type relationships for these other domains of energy propagation.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix, we provide details of our approach. First we state the Green's functions used in Eq. (1). Then we use a stationary-phase analysis to derive the contributing terms to the interferometric relationship and finally we derive a generalized optical theorem for surface waves.

Following Snieder [20], the single (point) scattered surface-wave mode σ, $u_i^{\sigma(1)}(\mathbf{r}_B,\omega)$, at a location $\mathbf{r}_B$ due to an incident surface-wave mode ν at scattering location $\mathbf{r}_0$, $u_i^{\nu(0)}(\mathbf{r}_0,\omega)$, generated by a point force in the m direction at location $\mathbf{r}_A$ is

$$u_i^{\sigma(1)}(\mathbf{r}_B,\omega) = \sum_{\sigma\nu} \frac{e^{i(k_\sigma X_{0B}+k_\nu X_{A0}+\pi/2)}}{\frac{\pi}{2}\sqrt{k_\nu k_\sigma X_{0B}X_{A0}}} p_i^{\sigma}(z_B,\varphi_{0B}) \times p_m^{\nu^*}(z_A,\varphi_{A0}) V^{\sigma\nu}(\varphi_{0B},\varphi_{A0}), \quad (15)$$

where $\mathbf{r_0}$ is the scattering location, $V^{\sigma\nu}(\varphi_{0B},\varphi_{A0})$ is the scattering matrix for an incident wave with azimuth φ_{A0} and a scattered wave with azimuth φ_{0B} (the depth of the scatterer z_0 is implicit in $V^{\sigma\nu}$), k_ν is the wave number associated with the νth surface-wave mode, X_{A0} and X_{0B} are the horizontal offsets between the scatterer at $\mathbf{r}_0$ and locations $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively, φ_{A0} and φ_{0B} are also the azimuth of the horizontal paths between $\mathbf{r}_A$ and $\mathbf{r}_0$, and $\mathbf{r}_0$ and $\mathbf{r}_B$, respectively, and z_A and z_B are the depths of $\mathbf{r}_A$ and $\mathbf{r}_B$, respectively (Fig. 3). Superscripts (0) and (1) refer to the wave field in the background medium and the scattered wave field, respectively. To simplify the expression, the modal normalization $8c^\nu U^\nu I_1^\nu=1$ is assumed, where c^ν, U^ν, and I_1^ν are the phase velocity, group velocity, and kinetic energy for the current mode, and p_i^ν is the ith component of the polarization vector,

$$\mathbf{p}^\nu(z,\varphi) = \begin{pmatrix} r_1^\nu(z)\cos\varphi \\ r_1^\nu(z)\sin\varphi \\ i r_2^\nu(z) \end{pmatrix}, \quad (16)$$

where $r_1^\nu(z)$ and $r_2^\nu(z)$ are the horizontal and vertical Rayleigh-wave eigenfunctions, respectively. This wave-field representation is for a single frequency, and in the following we assume summation over the relevant frequency range.

Mode ν of the incident surface wave due to the same source at location $\mathbf{r}_A$ is

$$u_i^{\nu(0)}(\mathbf{r}_B,\omega) = p_i^\nu(z_B,\varphi_{AB}) p_m^{\nu^*}(z_A,\varphi_{AB}) \frac{e^{i(k_\nu X_{AB}+\pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X_{AB}}}, \quad (17)$$

where X_{AB} and φ_{AB} are the offset and azimuth describing the horizontal projection of the path between $\mathbf{r}_A$ and $\mathbf{r}_B$ (Fig. 4).

The Green's function is then the sum of the direct and scattered surface wave,

$$G_{im}^{\sigma\nu}(\mathbf{r}_B,\mathbf{r}_A) = p_i^\nu(z_B,\varphi_{AB}) p_m^{\nu^*}(z_A,\varphi_{AB}) \times \frac{e^{i(k_\nu X_{AB}+\pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X_{AB}}} + \frac{e^{i(k_\sigma X_{0B}+k_\nu X_{A0}+\pi/2)}}{\frac{\pi}{2}\sqrt{k_\nu k_\sigma X_{0B}X_{A0}}} \times p_i^\sigma(z_B,\varphi_{0B}) p_m^{\nu^*}(z_A,\varphi_{A0}) V^{\sigma\nu}(\varphi_{0B},\varphi_{A0}). \quad (18)$$

In Eq. (3) of the main text, we simplify this expression by writing $G_{im}^{\sigma\nu}(\mathbf{r}_B,\mathbf{r}_A)=G_{im}^{0}(\mathbf{r}_B,\mathbf{r}_A)+G_{im}^{sc}(\mathbf{r}_B,\mathbf{r}_A)$, where $G_{im}^{0}(\mathbf{r}_B,\mathbf{r}_A)$ is the direct wave, and $G_{im}^{sc}(\mathbf{r}_B,\mathbf{r}_A)$ is the scattered wave. The equivalent particle-displacement deformation-rate Green's function is

$$n_j c_{njkm}\partial_k G_{im}^{\sigma\nu}(\mathbf{r}_B,\mathbf{r}_A) = p_i^\nu(z_B,\varphi_{AB}) T_m^{\nu^*}(z_A,\varphi_{AB}) \frac{e^{i(k_\nu X_{AB}+\pi/4)}}{\sqrt{\frac{\pi}{2}k_\nu X_{AB}}} + \frac{e^{i(k_\sigma X_{0B}+k_\nu X_{A0}+\pi/2)}}{\frac{\pi}{2}\sqrt{k_\nu k_\sigma X_{0B}X_{A0}}} p_i^\sigma(z_B,\varphi_{0B}) T_m^{\nu^*}(z_A,\varphi_{A0}) V^{\sigma\nu}(\varphi_{0B},\varphi_{A0}), \quad (19)$$

where T_n^ν is the nth component of the traction vector

$$\mathbf{T}^{\nu}(z,\varphi)=\begin{pmatrix} ik_\nu r_1^\nu(z)\cos^2\varphi & ik_\nu r_1^\nu(z)\cos\varphi\sin\varphi & -k_\nu r_2^\nu(z)\cos\varphi \\ ik_\nu r_1^\nu(z)\cos\varphi\sin\varphi & ik_\nu r_1^\nu(z)\sin^2\varphi & -k_\nu r_2^\nu(z)\sin\varphi \\ \frac{\partial}{\partial z}r_1^\nu(z)\cos\varphi & \frac{\partial}{\partial z}r_1^\nu(z)\sin\varphi & \frac{\partial}{\partial z}ir_2^\nu(z) \end{pmatrix} n_j c_{njkl}. \tag{20}$$

Equation (19) is simplified in Eq. (4) of the main text by writing $n_j c_{njkm}\partial_k G_{im}^{\sigma\nu}(\mathbf{r}_B,\mathbf{r}_A)=\partial G_{im}^{0}(\mathbf{r}_B,\mathbf{r}_A)+\partial G_{im}^{sc}(\mathbf{r}_B,\mathbf{r}_A)$.

While Eqs. (15)–(20) provide semianalytical representations of the Rayleigh-wave Green's functions, similar expressions exist for the Love-wave Green's functions, and for scattering conversions between Love-wave modes and Rayleigh-wave modes [20]. Hence we expect that our analysis also holds for the Love-wave case.

In the main text, we discuss the four terms that are introduced when we substitute expressions (18) and (19) into Eq. (1). Here we analyze each of these terms in turn (excluding $T1$ which has been the subject of a previous study [30]).

After substituting the appropriate Green's functions into Eq. (1) we find that the second term $T2$ is

$$T2=\int_S \frac{e^{i(-k_\nu X_{SA}+k_\sigma X_{0B}+k_\nu X_{S0}+\pi/4)}}{\frac{\pi}{2}\sqrt{\frac{\pi}{2}k_\sigma k_\nu k_\nu X_{SA}X_{0B}X_{S0}}} p_i^{\sigma}(z_B,\varphi_{0B})p_m^{\nu^*}(z_A,\varphi_{SA}) \times V^{\sigma\nu}(\varphi_{0B},\varphi_{S0})[p_n^{\nu^*}(z_S,\varphi_{S0})T_n^{\nu}(z_S,\varphi_{SA}) - p_n^{\nu}(z_S,\varphi_{SA})T_n^{\nu^*}(z_S,\varphi_{S0})]dS, \tag{21}$$

where the geometric variables are illustrated in Fig. 5. In order to analyze this integral, we use the cylindrical coordinate system introduced in the main text, and in Appendix B we find that the stationary-phase conditions are $\varphi_{S0}-\varphi_{A0}=0$ and $\varphi_{S0}-\varphi_{A0}=\pi$. In cylindrical coordinates, we have $dS=X_{S0}d\varphi_{S0}dz$. We use $\varphi_{S0}=\varphi_{SA}$ at the stationary point and follow Halliday and Curtis [30] who used the isotropic form of the stress tensor and solve the depth-dependant part of this integral using,

$$\int_0^\infty p_n^{\nu^*}(z_S,\varphi_{S0})T_n^{\nu}(z_S,\varphi_{S0})-p_n^{\nu}(z_S,\varphi_{S0})T_n^{\nu^*}(z_S,\varphi_{S0})dz = \frac{1}{2}ik_\nu(\cos\varphi_{S0}n_x+\sin\varphi_{S0}n_y). \tag{22}$$

Since we are evaluating this integral at the stationary point, the integrand in Eq. (22) is only dependant on depth (φ_{S0} is fixed at the stationary point). Note that the integral contains a sum over the indice n, i.e., we sum over the three components of the normal to the boundary S. Expression (22) greatly reduces the complexity of the problem and allows for the analysis of the integral using the method of stationary phase. If we allow the integration surface to be a cylinder with extremely large radius such that $\cos\varphi_{S0}=-n_x$ and $\sin\varphi_{S0}=-n_y$,

$$\int_0^\infty p_n^{\nu^*}(z_S,\varphi_{S0})T_n^{\nu}(z_S,\varphi_{S0})-p_n^{\nu}(z_S,\varphi_{S0})T_n^{\nu^*}(z_S,\varphi_{S0})dz=-\frac{1}{2}ik_\nu. \tag{23}$$

Using this relationship we find,

$$T2=-\frac{ik_\nu}{\pi}\int_R \frac{e^{i(-k_\nu X_{SA}+k_\sigma X_{0B}+k_\nu X_{S0}+\pi/4)}}{k_\nu\sqrt{\frac{\pi}{2}k_\sigma X_{SA}X_{0B}X_{S0}}} \times p_i^{\sigma}(z,\varphi_{0B})p_m^{\nu^*}(z,\varphi_{SA})V^{\sigma\nu}(\varphi_{0B},\varphi_{S0})X_{S0}d\varphi_{S0}. \tag{24}$$

The integration domain has been changed from the domain S to the domain R; this domain R represents the horizontal plane of integration described by $X_{S0}d\varphi_{S0}$. We now wish to solve the integral

$$I2=\int_R \frac{e^{i(-k_\nu X_{SA}+k_\sigma X_{0B}+k_\nu X_{S0}+\pi/4)}}{k_\nu\sqrt{\frac{\pi}{2}k_\sigma X_{SA}X_{0B}X_{S0}}}V^{\sigma\nu}(\varphi_{0B},\varphi_{S0})X_{S0}d\varphi_{S0}, \tag{25}$$

using the method of the stationary phase. This requires the second derivatives of X_{SA} and X_{S0},

$$\frac{\partial^2 X_{SA}}{\partial\varphi_{S0}^2}=\frac{X_{S0}X_{A0}\cos(\varphi_{S0}-\varphi_{A0})}{\sqrt{X_{S0}^2-2X_{S0}X_{A0}\cos(\varphi_{S0}-\varphi_{A0})+X_{A0}^2}}, \tag{26}$$

and

$$\frac{\partial^2 X_{S0}}{\partial\varphi_{S0}^2}=0. \tag{27}$$

At the first stationary point ($\varphi_{S0}-\varphi_{A0}=0$) Eq. (26) becomes,

$$\frac{\partial^2 X_{SA}}{\partial\varphi_{S0}^2}=\frac{X_{S0}X_{A0}}{X_{SA}}, \tag{28}$$

since at this stationary point $X_{SA}=X_{S0}-X_{A0}$ [Fig. 6(a)]. At the second stationary point $\varphi_{S0}-\varphi_{A0}=\pi$ and $X_{SA}=X_{S0}+X_{A0}$ [Fig. 6(b)] so,

$$\frac{\partial^2 X_{SA}}{\partial\varphi_{S0}^2}=\frac{-X_{S0}X_{A0}}{X_{SA}}. \tag{29}$$

We first evaluate the stationary point $\varphi_{S0}-\varphi_{A0}=0$. Following Snieder [31], the solution to the integral is

$$I2 = \frac{e^{i(-k_\nu X_{SA}+k_\sigma X_{0B}+k_\nu X_{S0}+\pi/4)}}{k_\nu\sqrt{\frac{\pi}{2}k_\sigma X_{SA}X_{0B}X_{S0}}} \times e^{-i\pi/4}\sqrt{\frac{2\pi}{k_\nu}}\frac{X_{S0}}{\sqrt{\frac{X_{S0}X_{A0}}{X_{SA}}}}V^{\sigma\nu}(\varphi_{0B},\varphi_{S0}), \tag{30}$$

$$= \frac{2}{k_\nu}\frac{e^{i(-k_\nu X_{SA}+k_\sigma X_{0B}+k_\nu X_{S0})}}{\sqrt{k_\sigma k_\nu X_{0B}X_{A0}}}V^{\sigma\nu}(\varphi_{0B},\varphi_{S0}). \tag{31}$$

Substituting $I2$ into Eq. (24) we obtain

$$T2_p = -\frac{2i}{\pi}\frac{e^{i(-k_\nu X_{SA}+k_\sigma X_{0B}+k_\nu X_{S0})}}{\sqrt{k_\sigma k_\nu (X_{0B}X_{A0})}} \times p_i^\sigma(z,\varphi_{0B})p_m^{\nu^*}(z,\varphi_{SA})V^{\sigma\nu}(\varphi_{0B},\varphi_{S0}), \tag{32}$$

where the subscript p indicates that this is a physical term, corresponding to the scattered term of Eq. (18). If $\varphi_{S0}-\varphi_{A0}=0$ then $X_{SA}=X_{S0}-X_{A0}$ and $\varphi_{S0}=\varphi_{SA}=\varphi_{A0}$, so the integral becomes

$$T2_p = -\frac{e^{i(k_\nu X_{A0}+k_\sigma X_{0B}+\pi/2)}}{\frac{\pi}{2}\sqrt{k_\sigma k_\nu (X_{0B}X_{A0})}}p_i^\sigma(z,\varphi_{0B}) \times p_m^{\nu^*}(z,\varphi_{A0})V^{\sigma\nu}(\varphi_{0B},\varphi_{A0}). \tag{33}$$

Thus term $T2$ provides the correct causal scattered surface wave as desired [cf. the second term of Eq. (18)]. Following a similar process for the second stationary point (when $\varphi_{S0}-\varphi_{A0}=\pi$, $X_{SA}=X_{S0}+X_{A0}$, and $\varphi_{S0}=\varphi_{SA}=\varphi_{A0}+\pi$) the integral becomes

$$T2_{np} = -\frac{e^{i(k_\sigma X_{0B}-k_\nu X_{A0})}}{\frac{\pi}{2}\sqrt{k_\sigma k_\nu (X_{A0}X_{0B})}}p_i^\sigma(z,\varphi_{0B}) \times p_m^{\nu^*}(z,\varphi_{A0}+\pi)V^{\sigma\nu}(\varphi_{0B},\varphi_{A0}+\pi). \tag{34}$$

This term does not correspond to any part of the Green's function defined in Eq. (18)—subscript np indicates that this is nonphysical. Note that if we reverse the order of cross correlation (i.e., use the direct surface wave at $\mathbf{r}_B$ and the scattered surface wave at $\mathbf{r}_A$) and repeat the above process to analyze contribution $T3$, we find that the two terms are equal to

$$T3_p = \frac{e^{-i(k_\nu X_{A0}+k_\sigma X_{0B}+\pi/2)}}{\frac{\pi}{2}\sqrt{k_\sigma k_\nu (X_{A0}X_{0B})}}p_i^{\sigma^*}(z,\varphi_{0B}) \times p_m^\nu(z,\varphi_{A0})V^{\sigma\nu^*}(\varphi_{0B},\varphi_{A0}), \tag{35}$$

and for the second term

$$T3_{np} = \frac{e^{i(k_\sigma X_{0B}-k_\nu X_{A0})}}{\frac{\pi}{2}\sqrt{k_\sigma k_\nu (X_{A0}X_{0B})}}p_i^{\sigma^*}(z,\varphi_{0B}) \times p_m^\nu(z,\varphi_{A0}+\pi)V^{\sigma\nu^*}(\varphi_{0B},\varphi_{A0}+\pi), \tag{36}$$

where we have used $V^{\nu\sigma}(\varphi_{A0},\varphi_{0B})=V^{\sigma\nu}(\varphi_{0B},\varphi_{A0})$. The geometries for these two terms are illustrated in Figs. 6(c) and 6(d). Again by comparing with Eq. (18), we see that $T3_p$ contributes the true scattered surface-wave event but in the time-reversed part of the interferometric integral due to the complex conjugation of Eq. (35) with respect to the second term in Eq. (18). $T3_{np}$ on the other hand contributes a nonphysical arrival with the same phase as $T2_{np}$ but with opposite sign and complex conjugation of the scattering matrix.

In the main text, we combine terms $T1$, $T2_p$, and $T3_p$ which gives the Green's function described by the left-hand side of Eq. (1). However, to satisfy Eq. (1) we require that the nonphysical terms $T2_{np}$ and $T3_{np}$ are canceled. In the remainder of this appendix, and in the main text, we show that term $T4$ allows for the cancellation of the nonphysical terms provided scattering is governed by a generalized optical theorem for surface waves.

Term $T4$ is the cross correlation of the scattered surface waves recorded at both receivers

$$T4 = \int_S \frac{e^{i(-k_\sigma X_{A0}-k_\nu X_{S0}+k_\sigma X_{0B}+k_\nu X_{S0})}}{\frac{\pi^2}{4}k_\sigma k_\nu\sqrt{X_{S0}X_{A0}X_{S0}X_{0B}}}p_i^\sigma(z,\varphi_{0B}) \times p_m^{\sigma^*}(z,\varphi_{A0}+\pi)V^{\sigma\nu}(\varphi_{0B},\varphi_{S0})V^{\sigma\nu^*}(\varphi_{A0}+\pi,\varphi_{S0}) \times[p_n^{\nu^*}(z_S,\varphi_{S0})T_n^\nu(z_S,\varphi_{S0})-p_n^\nu(z_S,\varphi_{S0})T_n^{\nu^*}(z_S,\varphi_{S0})]dS. \tag{37}$$

Note that the incident wave field upon the scatterer is the same for both receiver positions. Recalling Eq. (23) and by using $dS=dzdrd\varphi_{S0}$, we can solve the depth dependant part of the integral

$$T4 = -\frac{2i}{\pi^2}\int_R \frac{e^{i(-k_\sigma X_{A0}+k_\sigma X_{0B})}}{X_{S0}k_\sigma\sqrt{X_{A0}X_{0B}}}p_i^\sigma(z,\varphi_{0B})p_m^{\sigma^*}(z,\varphi_{A0}+\pi) \times V^{\sigma\nu}(\varphi_{0B},\varphi_{S0})V^{\sigma\nu^*}(\varphi_{A0}+\pi,\varphi_{S0})drd\varphi_{S0}. \tag{38}$$

Note again the change of integration domain from S to R; this domain R represents the horizontal plane of integration described by $drd\varphi_{S0}$. Since X_{A0} and X_{0B} are constant, this term is always stationary: each source location provides a contribution to the interferometric integral and no such contributions cancel destructively within the integration.

We have already shown that the correct direct and scattered surface waves are recovered from terms $T1$, $T2_p$, and $T3_p$. However nonphysical arrivals are introduced by terms $T2_{np}$, $T3_{np}$, and $T4$, and since Eq. (1) is exact these terms must cancel. We group the terms $T2_{np}$ and $T3_{np}$ into a single nonphysical term T_{np},

$$T_{np} = -\frac{2}{\pi} A^{\sigma\nu}(X_{0B},X_{A0})[P_{im}^{\sigma\nu}(\varphi_{0B},\varphi_{A0}+\pi)V^{\sigma\nu}(\varphi_{0B},\varphi_{A0}+\pi) - P_{im}^{\sigma\nu^*}(\varphi_{0B},\varphi_{A0}+\pi)V^{\sigma\nu^*}(\varphi_{0B},\varphi_{A0}+\pi)]. \quad (39)$$

The propagation characteristics (the phase, wave numbers, and geometrical spreading) of the nonphysical term are given by

$$A^{\sigma\nu} = \frac{e^{i(k_\sigma X_{0B} - k_\nu X_{A0})}}{\sqrt{k_\nu k_\sigma (X_{A0} X_{0B})}} \quad (40)$$

and the polarization of the nonphysical term is given by

$$P_{im}^{\sigma\nu}(\varphi_{0B},\varphi_{A0}+\pi) = p_i^{\sigma}(z,\varphi_{0B}) p_m^{\nu^*}(z,\varphi_{A0}+\pi). \quad (41)$$

We can therefore write term $T4$ in the following condensed form:

$$T4 = -\frac{2i}{\pi^2}\int_R [A^{\sigma\sigma}(X_{0B},X_{A0})P_{im}^{\sigma\sigma^*}(\varphi_{0B},\varphi_{A0}+\pi) \times V^{\sigma\nu}(\varphi_{0B},\varphi_{S0})V^{\sigma\nu^*}(\varphi_{A0}+\pi,\varphi_{S0})]\frac{1}{X_{S0}} dr d\varphi_{S0}, \quad (42)$$

and since X_{S0} is equal to the radius of the cylinder,

$$T4 = -\frac{2i}{\pi^2}\int_0^{2\pi} [A^{\sigma\sigma}(X_{0B},X_{A0})P_{im}^{\sigma\sigma^*}(\varphi_{0B},\varphi_{A0}+\pi) \times V^{\sigma\nu}(\varphi_{0B},\varphi_{S0})V^{\sigma\nu^*}(\varphi_{A0}+\pi,\varphi_{S0})] d\varphi_{S0}. \quad (43)$$

In the generalized optical theorem, the scattering amplitude is often defined in terms of unit vectors in the directions of propagation of the incident and scattered waves. Following this convention, we take the horizontal component of the unit vectors (implicit in the following notation) and recalling our vector definition (2) we redefine the scattering matrix and polarization terms as

$$V^{\sigma\nu}(\varphi_{0B},\varphi_{A0}+\pi) = f^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A), \quad (44)$$

and

$$P_{im}^{\sigma\nu}(\varphi_{0B},\varphi_{A0}+\pi) = P_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A), \quad (45)$$

where $f^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)$ is the surface-wave scattering matrix for an incident surface-wave mode ν traveling in the horizontal direction of unit vector $\hat{\mathbf{r}}_A$ that is scattered in the horizontal direction of unit vector $\hat{\mathbf{r}}_B$ as surface-wave mode σ. In the main text, we combine these two terms using $S_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A) = P_{im}^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A) f^{\sigma\nu}(\hat{\mathbf{r}}_B,\hat{\mathbf{r}}_A)$, where it is understood that $S_{im}^{\sigma\nu}$ includes a combination of the polarization components i and m. We also replace φ_{S0} with Ω and then Eqs. (39) and (43) are rewritten to reach Eqs. (7) and (9) of the main text, from which we derive a generalized optical theorem for surface waves.

Finally, in Eq. (13) of the main text we use a term $D^{\sigma\nu}$ which accounts for the differences in phase and wave number of the observed scattered surface-wave modes σ and ν (i.e., the terms that describe the propagation characteristics of each mode). This term is defined as

$$D^{\sigma\nu} = \frac{A^{\sigma\sigma}}{A^{\sigma\nu}}, \quad (46)$$

$$= e^{i(k_\nu X_{A0} - k_\sigma X_{A0})}\sqrt{\frac{k_\nu}{k_\sigma}}. \quad (47)$$

APPENDIX B

In this appendix, we find the stationary-phase condition for the scattered surface waves. To find the stationary-phase condition, we need the lengths of each of the propagation paths. In cylindrical coordinates, the length X_{SA} can be related to the other paths as follows:

$$X_{SA} = \sqrt{X_{S0}^2 - 2X_{S0}X_{A0}\cos(\varphi_{S0}-\varphi_{A0}) + X_{A0}^2}, \quad (48)$$

where geometric variables are illustrated in (Fig. 2). In order to determine the stationary points of the integral, we then require the first derivatives of X_{SA}, X_{S0}, X_{A0}, and X_{0B} with respect to the integration direction. Since there is no dependence on z, we consider the φ_{S0} derivatives using the geometry defined in Eq. (2)

$$\frac{\partial X_{SA}}{\partial \varphi_{S0}} = 2X_{S0}X_{A0}\sin(\varphi_{S0}-\varphi_{A0}), \quad (49)$$

$$\frac{\partial X_{A0}}{\partial \varphi_{S0}} = 0, \quad (50)$$

$$\frac{\partial X_{0B}}{\partial \varphi_{S0}} = 0, \quad (51)$$

$$\frac{\partial X_{S0}}{\partial \varphi_{S0}} = 0. \quad (52)$$

In our analysis, we require stationary-phase conditions for integration in the φ_{S0} direction. The integral $T2$ is stationary when

$$\frac{\partial X_{S0}}{\partial \varphi_{S0}} = \frac{\partial X_{SA}}{\partial \varphi_{S0}}, \quad (53)$$

i.e.,

$$0 = 2X_{S0}X_{A0}\sin(\varphi_{S0}-\varphi_{A0}) = 2X_{S0}X_{A0}\sin(\varphi_{S0}-\varphi_{A0}) \quad (54)$$

i.e., the stationary-phase conditions are $(\varphi_{S0}-\varphi_{A0})=0$ and $(\varphi_{S0}-\varphi_{A0})=\pi$.

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